Math 429 - Exercise Sheet 8

1. Calculate the Killing form of $\mathfrak{sl}_n$.

2. Let $\mathfrak{p} \subset \mathfrak{sl}_{m+n}$ be the **parabolic** subalgebra consisting of matrices of the form

$$\begin{pmatrix} A & X \\ 0 & B \end{pmatrix}$$

where A and B are traceless $m \times m$ and $n \times n$, respectively. Calculate $\mathrm{rad}(\mathfrak{p})$ and $\mathfrak{p}_{ss} = \mathfrak{p}/\mathrm{rad}(\mathfrak{p})$.

3. Calculate the Casimir element of $\mathfrak{o}_3$ and its action on the tautological 3-dimensional representation of $\mathfrak{o}_3$.

4. If X is a diagonalizable $n \times n$ matrix, prove that

$$\mathrm{ad}_X : \mathfrak{gl}_n \to \mathfrak{gl}_n, \qquad \mathrm{ad}_X(Y) = [X, Y]$$

is also diagonalizable. The corresponding claim with "diagonalizable" replaced by "semisimple" requires a little bit of field theory, so we won't touch upon this (but if you want a challenge, show that a matrix over an algebraically closed field is diagonalizable if and only if it is semisimple).

5. If we assume that a $n \times n$ complex matrix X is conjugate to a direct sum of Jordan blocks, then

- explicitly construct a diagonalizable matrix X_{ss} and a nilpotent matrix X_n such that

$$X = X_{ss} + X_n \tag{1}$$

- show that $X_{ss}X_n = X_nX_{ss}$
- show X_{ss} and X_n are complex polynomials in X with zero constant term
- show that the decomposition (1) is unique with respect to the properties above.

(*) Prove the following analogue of the claim at the beginning of the proof of Theorem 17. For any Lie algebra $\mathfrak{g}$, consider its Lie algebra of derivations

$$\mathrm{Der}(\mathfrak{g}) \subseteq \mathrm{End}(\mathfrak{g})$$

as in Subsection 8.7. Show that the semisimple and nilpotent part of any $\zeta \in \mathrm{Der}(\mathfrak{g})$ (calculated as linear transformations of $\mathfrak{g}$) also lie in $\mathrm{Der}(\mathfrak{g})$.